On a Form of the Solution of Laplace's Equation Suitable for Problems Relating to Two Spheres.

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§ 1. *Introduction.*

The problems presented by the motion of two solid spheres in a perfect fluid have been attacked by various writers.* In each case the method has been that of approximation by successive images, and it appears that no general analytical method of solution has been developed as in the case of the analogous problems for the sphere, ellipsoid and anchor-ring. In this paper a general solution of Laplace's equation is obtained in a form suitable for problems in which the boundary conditions are given over any two spherical surfaces. A similar solution is obtained of the differential equation of Stokes' current function. With the aid of these results it is theoretically possible to determine completely a potential function when its value is specified over any two spheres. The method is illustrated by an application to the electrostatic field of two charged conducting spheres. In this case the method leads to a simple expression for the capacity of either of the spheres.

The co-ordinates employed are defined by rotating about the z axis the system of circles, in any plane, through two fixed points on the axis and the orthogonal system of circles. Thus, if x, y, z, are the Cartesian co-ordinates and $\rho = \sqrt{(x^2+y^2)}$, and the distance between the fixed points is $2a$, we have a system of orthogonal curvilinear co-ordinates u, v, w, where

$$u+iv = \log\frac{\rho+i(z+a)}{\rho+i(z-a)}, \qquad w = \tan^{-1}\frac{y}{x}. \tag{1}$$

The surfaces u = constant will then be a series of coaxal spheres having a common radical plane $u = 0$. It is obvious that the origin and the value of a can be so chosen that any two given spheres will be included in the system. These co-ordinates are similar to those employed by Hicks in his memoir on "Toroidal Functions,"† the difference being that in the present

* R. A. Herman, "On the Motion of Two Spheres in a Fluid," 'Quart. Journ. of Maths.,' 1887, vol. 22; W. M. Hicks, "On the Motion of Two Spheres in a Fluid," 'Phil. Trans.,' 1880; A. B. Basset, "On the Motion of Two Spheres in a Liquid," 'Proc. Lond. Math. Soc.,' 1887.

† 'Phil. Trans.,' 1881, p. 609.

case the circles are rotated about the line through the limiting points instead of about their common radical axis. Further, in one case the surface conditions are given over spheres, while in the other they are given over "tores" or anchor-rings. For these reasons, although that part of Hicks' paper which deals with the general theory of curvilinear co-ordinates has been of very considerable help to the present writer, the functions obtained and the methods suitable for their applications are very different in the two cases. The following results will be of use; they are easily obtained from (1) and are set down here for reference:—

$$\rho = \frac{a \sin v}{\cosh u - \cos v}, \qquad z = \frac{a \sinh u}{\cosh u - \cos v}. \tag{2}$$

$$\left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left\{\left(\frac{\partial \rho}{\partial u}\right)^2 + \left(\frac{\partial \rho}{\partial v}\right)^2\right\}^{-1} = \frac{(\cosh u - \cos v)^2}{a^2}. \tag{3}$$

If δs_u, δs_v denote the lengths of elements of the curves $u =$ constant and $v =$ constant respectively,

$$\left.\begin{aligned} \delta s_u &= \left\{\left(\frac{\partial \rho}{\partial u}\right)^2 + \left(\frac{\partial \rho}{\partial v}\right)^2\right\}^{\frac{1}{2}} \delta v = \frac{a}{\cosh u - \cos v} \delta v. \\ \delta s_v &= \left\{\left(\frac{\partial \rho}{\partial u}\right)^2 + \left(\frac{\partial \rho}{\partial v}\right)^2\right\}^{\frac{1}{2}} \delta u = \frac{a}{\cosh u - \cos v} \delta u. \end{aligned}\right\} \tag{4}$$

If r be the radius of any sphere of the system, and d the distance of its centre from the origin,

$$r = a/|\sinh u|, \qquad d = a \coth u. \tag{5}$$

§ 2. *Solution of* $\nabla^2\phi = 0$.

It is well known that if u, v, w, be any system of orthogonal co-ordinates Laplace's equation may be written in the form

$$\nabla^2\phi \equiv h_1 h_2 h_3 \left[\frac{\partial}{\partial u}\left(\frac{h_1}{h_2 h_3}\frac{\partial \phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_2}{h_3 h_1}\frac{\partial \phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_3}{h_1 h_2}\frac{\partial \phi}{\partial w}\right)\right] = 0, \tag{6}$$

where

$$h_1^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2,$$

$$h_2^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2,$$

$$h_3^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2.$$

In the present case, u, v, being conjugate functions of ρ, z,

$$h_1^2 = \left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial v}{\partial \rho}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 = h_2^2,$$

and if

$$w = \tan^{-1}\frac{y}{x}, \qquad h_3^2 = \frac{1}{\rho^2}.$$

Substituting these values in (6), we have, as the form taken by Laplace's equation in u, v, w co-ordinates,

$$\frac{\partial}{\partial u}\left(\rho\frac{\partial\phi}{\partial u}\right)+\frac{\partial}{\partial v}\left(\rho\frac{\partial\phi}{\partial v}\right)+\rho^{-1}\left\{\left(\frac{\partial u}{\partial\rho}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right\}^{-1}\frac{\partial^2\phi}{\partial w^2}=0. \tag{7}$$

In order to solve this equation, write $\phi = \psi/\sqrt{\rho}$. We obtain

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+\frac{\psi}{4\rho^2}\left\{\left(\frac{\partial\rho}{\partial u}\right)^2+\left(\frac{\partial\rho}{\partial v}\right)^2\right\}+\rho^{-2}\left\{\left(\frac{\partial u}{\partial\rho}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right\}^{-1}\frac{\partial^2\psi}{\partial w^2}=0,$$

which becomes, in virtue of equations (2) and (3),

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+\frac{1}{\sin^2 v}\left(\frac{\psi}{4}+\frac{\partial^2\psi}{\partial w^2}\right)=0. \tag{8}$$

After the usual manner we will seek a solution of the type $\psi = \mathrm{U, V, W}$, where U, V, W, are functions of u, v, w, respectively. Substituting in (8) we at once obtain

$$\mathrm{W} = \cos(mw+\alpha)$$

$$-\frac{1}{\mathrm{U}}\frac{d^2\mathrm{U}}{du^2}=\frac{1}{\mathrm{V}}\frac{d^2\mathrm{V}}{dv^2}+\frac{1-4m^2}{4\sin^2 v} = \text{const.} = -(n+\tfrac{1}{2})^2, \text{ say.}$$

Hence $$\mathrm{U} = \mathrm{A}\cosh(n+\tfrac{1}{2})u+\mathrm{B}\sinh(n+\tfrac{1}{2})u,$$

and $$\frac{d^2\mathrm{V}}{dv^2}+\left((n+\tfrac{1}{2})^2-\frac{4m^2-1}{4\sin^2 v}\right)\mathrm{V}=0.$$

Put $\mathrm{V} = \sqrt{(\sin v)}\,\chi$ and we obtain

$$\frac{d^2\chi}{dv^2}+\cot v\frac{d\chi}{dv}+\left(n^2+n-\frac{m^2}{\sin^2 v}\right)\chi=0.$$

Finally, writing $\cos v = \mu$, we have

$$(1-\mu^2)\frac{d^2\chi}{d\mu^2}-2\mu\frac{d\chi}{d\mu}+n(n+1)\chi=\frac{m^2}{1-\mu^2}\chi,$$

the solution of which is well known to be

$$\chi = a\mathrm{P}_n{}^m(\mu)+b\mathrm{Q}_n{}^m(\mu),$$

where $$\mathrm{P}_n{}^m(\mu) = (1-\mu^2)^{m/2}\frac{d^m\mathrm{P}_n(\mu)}{d\mu^m},$$

$\mathrm{P}_n(\mu)$ being the Legendre function of order n. $\mathrm{Q}_n{}^m(\mu)$ is the corresponding function of the "second kind." Hence

$$\mathrm{V} = \sqrt{(\sin v)}\,\{a\mathrm{P}_n{}^m(\mu)+b\mathrm{Q}_n{}^m(\mu)\},$$

and a particular solution of (7) may be expressed in the form

$$\phi = \sqrt{(\cosh u-\mu)}\cos(mw+\alpha)\{\mathrm{A}\cosh(n+\tfrac{1}{2})u+\mathrm{B}\sinh(n+\tfrac{1}{2})u\}\{a\mathrm{P}_n{}^m(\mu)+b\mathrm{Q}_n{}^m(\mu)\}.$$

We shall find that for physical applications it is sufficient to confine our

attention to integral values of n. Moreover, the solution corresponding to $n = -(\nu+1)$ is identical in form with that corresponding to $n = +\nu$. It will, therefore, be sufficient to consider only positive integral values of n, and since by the nature of the function $P_n^m(\mu)$, $m \leqq |n|$, we may write the solution of (7) in the form

$$\phi = \sqrt{(\cosh u - \mu)} \sum_{m=0}^{m=\infty} \sum_{n=m}^{n=\infty} \cos(mw+\alpha) \{A_{m,n} \cosh(n+\tfrac{1}{2})u + B_{m,n} \sinh(n+\tfrac{1}{2})u\} \{a_{m,n} P_n^m(\mu) + b_{m,n} Q_n^m(\mu)\}. \quad (9)$$

§ 3. *The Properties of the Function* ϕ.

The function ϕ and its first differential coefficients must be finite and continuous at all points of the field except those which correspond to some special physical condition such as a source or charge. $P_n^m(\cos v)$ is finite and continuous for all real values of v, but $Q_n^m(\cos v)$ becomes infinite when $v = 0$ or π, and hence cannot occur in an expression for ϕ which is to hold throughout a region including any part of the axis of z. It is to such cases that we confine our attention in the present paper. At the limiting points of the system u becomes infinite, and hence the forms of ϕ appropriate to regions including the points $u = {+\infty \atop -\infty}$ are respectively

$$\phi = \sqrt{(\cosh u - \mu)} \sum_{m=0}^{m=\infty} \sum_{n=m}^{n=\infty} A_{m,n} \cos(mw+\alpha) e^{\mp(n+\frac{1}{2})u} P_n^m(\mu).$$

It will be convenient for reference to have the general value of $\partial\phi/\partial u$. Differentiating (9) with respect to u we have, omitting the Q terms,

$$\begin{aligned}\frac{\partial\phi}{\partial u} = \frac{1}{2\sqrt{(\cosh u - \mu)}} \sum_{m=0}^{m=\infty} \sum_{n=m}^{n=\infty} \cos(mw+\alpha) \{&A_{m,n} \sinh u \cosh(n+\tfrac{1}{2})u \\ &+ B_{m,n} \sinh u \sinh(n+\tfrac{1}{2})u + (2n+1) A_{m,n} \cosh u \sinh(n+\tfrac{1}{2})u \\ &+ (2n+1) B_{m,n} \cosh u \cosh(n+\tfrac{1}{2})u \\ &- (2n+1)\mu [A_{m,n} \sinh(n+\tfrac{1}{2})u + B_{m,n} \cosh(n+\tfrac{1}{2})u]\} P_n^m(\mu).\end{aligned}$$

Applying the recurrence relation

$$(2n+1)\mu P_n^m(\mu) = (n+m) P_{n-1}^m(\mu) + (n-m+1) P_{n+1}^m(\mu),$$

we have, after further reduction,

$$\begin{aligned}\frac{\partial\phi}{\partial u} = \tfrac{1}{2}(\cosh u - \mu)^{-\frac{1}{2}} \sum_{m=0}^{m=\infty} \sum_{n=m}^{n=\infty} \cos(mw+\alpha) \{&[(n+1) A_{m,n} - (n+m+1) A_{m,n+1}] \\ &\times \sinh(n+\tfrac{3}{2})u + [nA_{m,n} - (n-m) A_{m,n-1}] \sinh(n-\tfrac{1}{2})u \\ &+ [(n+1) B_{m,n} - (n+m+1) B_{m,n+1}] \cosh(n+\tfrac{3}{2})u \\ &+ [nB_{m,n} - (n-m) B_{m,n-1}] \cosh(n-\tfrac{1}{2})u\} P_n^m(\mu), \end{aligned} \quad (10)$$

provided that (9) may be differentiated term by term and rearranged.

The coefficients in the function ϕ are not in all cases identical in the two regions into which the field is divided by the plane $u = 0$. Let (9) be the expression for ϕ in that part of the field for which $u > 0$ while a similar expression with accented coefficients holds throughout the other half of the field. In order that ϕ may be continuous across the plane $u = 0$

$$A'_{m,n} = A_{m,n},$$

while, in order that $\partial\phi/\partial u$ may be continuous, we must have (provided that $\partial\phi/\partial u$ is given by (10))

$$(n-1)\,B_{m,n}-(n+m+1)\,B_{m,n+1}+nB_{m,n}-(n-m)\,B_{m,n-1}$$
$$= (n+1)\,B'_{m,n}-(n+m+1)\,B'_{m,n+1}+nB'_{m,n}-(n-m)\,B'_{m,n-1}$$

where $n \geqq m$ and $B_{m,n} = 0$ if $n < m$.

$$(n+m+1)\,(B_{m,n}-B_{m,n+1})-(n-m)\,(B_{m,n-1}-B_{m,n})$$
$$= (n+m+1)\,(B'_{m,n}-B'_{m,n+1})-(n-m)\,(B'_{m,n-1}-B'_{m,n}).$$

Writing $\beta_{m,n} = B_{m,n}-B_{m,n-1}$,

$$(n+m+1)\,(\beta_{m,n+1}-\beta'_{m,n+1}) = (n-m)\,(\beta_{m,n}-\beta'_{m,n}).$$

This relation holds for all values of n such that $n \geqq m$. Hence

$$\beta_{m,n}-\beta'_{m,n} = 0 \quad \text{if} \quad n \geqq m+1,$$

which at once leads to the result

$$B_{m,n}-B'_{m,n} = B_{m,m}-B'_{m,m}.$$

These forms occur whenever the field includes *both* limiting points together with any part of the plane $u = 0$, for any expression for ϕ of the form (9) which is finite at one of these points becomes infinite at the other.

For a unit charge at the points $u = {+\infty \atop -\infty}$ the potential is given by

$$\phi = \{\rho^2+(z\mp a)^2\}^{-\frac{1}{2}} = \frac{1}{\sqrt{2}\,.\,a}\sqrt{(\cosh u-\mu)}\,e^{\pm u/2}.$$

Thus, as in the case of spherical harmonics, the harmonic of zero order corresponds to a point charge. The analogy, however, extends no further; the nth harmonic does not correspond to a charge of the nth order. It may be shown that, if r_1 be the radius vector from the pole $u = +\infty$,

$$\frac{2^{(n+\frac{1}{2})}a^{(n+1)}}{n\,!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r_1}\right) = \sqrt{(\cosh(u-\mu)}\,[e^{\frac{1}{2}u}-{}^nC_1e^{\frac{3}{2}u}P_1(\mu)+{}^nC_2e^{\frac{5}{2}u}P_2(\mu)$$
$$+\ldots+(-1)^r\,{}^nC_re^{(r+\frac{1}{2})u}P_r(\mu)+\ldots]$$

where ${}^nC_r$ is the ordinary binomial coefficient.

§ 4. *Case of Symmetry about an Axis; Stokes' Current Function.*

When the field is symmetrical about the axis of z, we obtain the form of the function ϕ by putting $m = 0$ in (9)

$$\phi = \sqrt{(\cosh u - \mu)} \sum_{n=0}^{n=\infty} \{A_n \cosh (n+\tfrac{1}{2})\, u + B_n \sinh (n+\tfrac{1}{2})\, u\}\, P_n(\mu). \tag{11}$$

Many physical problems are more easily interpreted by means of "lines of force," or "lines of flow," than by the potential. In the symmetrical case there exists a function ψ corresponding to Stokes' current function, such that the surfaces $\psi =$ constant are orthogonal to the equipotential surfaces, and which is connected with ϕ by the following relations*

$$\frac{\partial \psi}{\partial \rho} = \rho \frac{\partial \phi}{\partial z}; \qquad \frac{\partial \psi}{\partial z} = -\rho \frac{\partial \phi}{\partial \rho}.$$

Transforming to u, v co-ordinates,

$$\left(\frac{\partial \psi}{\partial u}\frac{\partial u}{\partial \rho} + \frac{\partial \psi}{\partial v}\frac{\partial v}{\partial \rho}\right) = \rho\left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z}\right),$$

$$\left(\frac{\partial \psi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \psi}{\partial v}\frac{\partial v}{\partial z}\right) = -\rho\left(\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial \rho} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial \rho}\right).$$

By a well-known property of conjugate functions $\dfrac{\partial u}{\partial \rho} = \dfrac{\partial v}{\partial z}$ and $\dfrac{\partial u}{\partial z} = -\dfrac{\partial v}{\partial \rho}$. Hence

$$\frac{\partial u}{\partial \rho}\left(\frac{\partial \psi}{\partial u} - \rho\frac{\partial \phi}{\partial v}\right) - \frac{\partial u}{\partial z}\left(\frac{\partial \psi}{\partial v} + \rho\frac{\partial \phi}{\partial u}\right) = 0,$$

$$\frac{\partial u}{\partial z}\left(\frac{\partial \psi}{\partial u} - \rho\frac{\partial \phi}{\partial v}\right) + \frac{\partial u}{\partial \rho}\left(\frac{\partial \psi}{\partial v} + \rho\frac{\partial \phi}{\partial u}\right) = 0.$$

Hence, since

$$\left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \neq 0,$$

$$\frac{\partial \psi}{\partial u} = \rho\frac{\partial \phi}{\partial v}; \qquad \frac{\partial \psi}{\partial v} = -\rho\frac{\partial \phi}{\partial u}. \tag{12}$$

When a potential exists, and is continuous, we may write

$$\frac{\partial}{\partial u}\left(\frac{\partial \phi}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial \phi}{\partial u}\right) = 0. \tag{13}$$

Substituting from (12), we obtain the differential equation satisfied by ψ,

$$\frac{\partial}{\partial u}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial v}\right) = 0,$$

an equation which is somewhat similar in form to Laplace's equation and

* Lamb, 'Hydrodynamics,' p. 117.

which yields to an analogous mode of treatment. Writing $\psi = \sqrt{\rho}\,.\,\Pi$, we have

$$\frac{\partial^2\Pi}{\partial u^2}+\frac{\partial^2\Pi}{\partial v^2}+\frac{\Pi}{2\rho}\left(\frac{\partial^2\rho}{\partial u^2}+\frac{\partial^2\rho}{\partial v^2}\right)-\frac{3\,\Pi}{4\rho^2}\left[\left(\frac{\partial\rho}{\partial u}\right)^2+\left(\frac{\partial\rho}{\partial v}\right)^2\right] = 0,$$

which becomes in the case under consideration

$$\frac{\partial^2\Pi}{\partial u^2}+\frac{\partial^2\Pi}{\partial v^2}-\frac{3\,\Pi}{4\sin^2 v} = 0.$$

Assuming, as before, a solution of the form $\Pi = \mathrm{UV}$,

$$\frac{1}{\mathrm{U}}\frac{d^2\mathrm{U}}{du^2} = -\frac{1}{\mathrm{V}}\frac{d^2\mathrm{V}}{dv^2}+\frac{3}{4\sin^2 v} = \text{constant} = (n+\tfrac{1}{2})^2.$$

Hence $$\mathrm{U} = a\cosh(n+\tfrac{1}{2})\,u+b\sinh(n+\tfrac{1}{2})\,u.$$

In order to solve for V, substitute $\mathrm{V} = \dfrac{\chi}{\sqrt{\sin v}}$. We obtain

$$\frac{d^2\chi}{dv^2}-\cot v\frac{d\chi}{dv}+n(n+1)\,\chi = 0.$$

Now write $\mu = \cos v$ and the equation becomes

$$(1-\mu^2)\frac{d^2\chi}{d\mu^2}+n(n+1)\,\chi = 0,$$

the general solution of which is

$$\chi = \mathrm{A}\,\{\mathrm{P}_{n-1}(\mu)-\mathrm{P}_{n+1}(\mu)\}+\mathrm{B}\,\{\mathrm{Q}_{n-1}(\mu)-\mathrm{Q}_{n+1}(\mu)\}.$$

As for the function ϕ, it is only necessary to consider positive integral values of n, and the solution of (13) which is finite on the axis is

$$\psi = \frac{1}{\sqrt{(\cosh u-\mu)}}\sum_{n=0}^{n=\infty}\{a_n\cosh(n+\tfrac{1}{2})\,u+b_n\sinh(n+\tfrac{1}{2})\,u\}\{\mathrm{P}_{n-1}(\mu)-\mathrm{P}_{n+1}(\mu)\}. \tag{14}$$

§ 5. *Properties of the Function* ψ.

It is obvious that the remarks made in § 3 on the continuity of the function ϕ apply equally to the function ψ. Differentiating (14) with respect to u and applying the recurrence relation

$$\mu\,[\mathrm{P}_{n-1}(\mu)-\mathrm{P}_{n+1}(\mu)] = \frac{n-1}{2n-1}[\mathrm{P}_{n-2}(\mu)-\mathrm{P}_n(\mu)]+\frac{n+2}{2n+3}[\mathrm{P}_n(\mu)-\mathrm{P}_{n+2}(\mu)],$$

we have

$$\begin{aligned}\frac{\partial\psi}{\partial u} = \frac{1}{2(\cosh u-\mu)^{\frac{3}{2}}}\sum_{n=0}^{n=\infty}\frac{n(n+1)}{2n+1}\Big\{&\frac{1}{n+1}[(2n+1)\,a_n-(2n+3)\,a_{n+1}]\\ &\qquad\times\sinh(n+\tfrac{3}{2})\,u\\ &-\frac{1}{n}[(2n-1)\,a_{n-1}-(2n+1)\,a_n]\sinh(n-\tfrac{1}{2})\,u\\ &+\frac{1}{n+1}[(2n+1)\,b_n-(2n+3)\,b_{n+1}]\cosh(n+\tfrac{3}{2})\,u\\ &-\frac{1}{n}[(2n-1)\,b_{n-1}-(2n+1)\,b_n]\cosh(n-\tfrac{1}{2})\,u\Big\}\,[\mathrm{P}_{n-1}(\mu)-\mathrm{P}_{n+1}(\mu)].\end{aligned} \tag{15}$$

As for ϕ the coefficients a must be the same on both sides of the plane $u = 0$, while to secure the continuity of the potential gradient we must have provided that (14) may be differentiated term by term

$$\frac{1}{n+1}(\beta_{n+1}-\beta_n)-\frac{1}{n}(\beta_n-\beta_{n-1}) = \frac{1}{n+1}(\beta_{n+1}'-\beta_n')-\frac{1}{n}(\beta_n'-\beta_{n-1}'),$$

where $\beta_n \equiv (2n+1)\,b_n$.

Putting $n = 1, 2, 3, \ldots, n-1$, and adding, we have

$$\frac{1}{n}(\beta_n-\beta_{n-1}-\beta_n'+\beta_{n-1}') = \beta_1-\beta_0+\beta_1'-\beta_0' = 2c,$$

where c is a constant.

Multiplying by n and again putting $n = 1, 2, \ldots, n$, we obtain on adding

$$\beta_n-\beta_n' = \beta_0-\beta_0'+2c\sum_1^n n,$$

or

$$b_n - b_n' = \frac{b_0-b_0'}{2n+1}+\frac{cn(n+1)}{2n+1}.$$

A simple example of this form is afforded by the case of a uniform field parallel to the axis of z in which

$$b_n-b_n' = \frac{2n(n+1)}{2n+1}.$$

Relations between the Coefficients in ϕ and ψ.

Let

$$\phi = (\cosh u-\mu)^{\frac{1}{2}}\, \Sigma\, \{A_n \cosh(n+\tfrac{1}{2})\,u+B_n \sinh(n+\tfrac{1}{2})\,u\}\, P_n(\mu),$$

$$\psi = (\cosh u-\mu)^{-\frac{1}{2}}\, \Sigma\, \{a_n \cosh(n+\tfrac{1}{2})\,u+b_n \sinh(n+\tfrac{1}{2})\,u\}\, \{P_{n-1}(\mu)-P_{n+1}(\mu)\},$$

be functions specifying the same field.

The coefficients will be related in such a way as to satisfy equations (12), which for our present purpose may be written in the form

$$\frac{\partial\psi}{\partial u} = \frac{a(\mu^2-1)}{\cosh u-\mu}\frac{\partial\phi}{\partial\mu}; \qquad \frac{\partial\psi}{\partial\mu} = \frac{a}{\cosh u-\mu}\frac{\partial\phi}{\partial u}. \tag{16}$$

Differentiating the expression for ϕ with respect to μ and applying the well-known recurrence relations we obtain

$$\frac{a(\mu^2-1)}{\cosh u-\mu}\frac{\partial\phi}{\partial\mu} = -\tfrac{1}{2}a(\cosh u-\mu)^{-\frac{1}{2}}\,\Sigma\Big[\frac{n(n+1)}{2n+1}\{(A_n-A_{n+1})\cosh(n+\tfrac{3}{2})\,u$$
$$+(A_n-A_{n-1})\cosh(n-\tfrac{1}{2})\,u+(B_n-B_{n+1})\sinh(n+\tfrac{3}{2})\,u$$
$$+(B_n-B_{n-1})\sinh(n-\tfrac{1}{2})\,u\}\,\{P_{n-1}(\mu)-P_{n+1}(\mu)\}\Big].$$

Identifying this expression with the value of $\partial\psi/\partial u$ given by (15), we have

$$\left.\begin{aligned}(2n+1)\,a_n-(2n-1)\,a_{n-1} &= -an(B_n-B_{n-1}),\\ (2n+1)\,b_n-(2n-1)\,b_{n-1} &= -an(A_n-A_{n-1}).\end{aligned}\right\} \tag{17}$$

It is easily verified that these are also the relations necessary and sufficient to satisfy the second of equations (12). Putting $n = 1, 2, 3, \ldots, n$, and adding, we obtain

$$\left.\begin{aligned}(2n+1)\,a_n &= -an\mathrm{B}_n + a\sum_0^{n-1}\mathrm{B}_n,\\ (2n+1)\,b_n &= -an\mathrm{A}_n + a\sum_0^{n-1}\mathrm{A}_n,\end{aligned}\right\} \tag{17'}$$

§ 6. *The Potential and Capacity of two Spherical Conductors.*

We will conclude the present paper by an application of this method to a classical electrostatic problem. Let u_1, u_2 be any two spheres such that $u_1 > 0$ but u_2 is unrestricted. The potential ϕ is constant over each of these spheres, and we can without loss of generality suppose it to be zero over the surface u_2 and V over the surface u_1. It is obvious that ϕ will be of the form

$$\phi = (\cosh u - \mu)^{\frac{1}{2}}\,\Sigma \mathrm{A}_n \sinh (n+\tfrac{1}{2})(u-u_2)\,\mathrm{P}_n(\mu),$$

from which we have

$$\mathrm{V}\,(\cosh u-\mu)^{-\frac{1}{2}} = \Sigma \mathrm{A}_n \sinh (n+\tfrac{1}{2})\,(u_1-u_2)\,\mathrm{P}_n(\mu).$$

The left-hand side may be written

$$\sqrt{2}\,.\,\mathrm{V}e^{-\frac{1}{2}u_1}(1-2\mu e^{-u_1}+e^{-2u_1})^{-\frac{1}{2}} = \sqrt{2}\,.\,\mathrm{V}\sum_{n=0}^{n=\infty} e^{-(n+\frac{1}{2})u_1}\mathrm{P}_n(\mu),$$

since $u_1 > 0$.

Equating the coefficients of $\mathrm{P}_n(\mu)$, we obtain the value of A_n, and the potential function becomes

$$\phi = \sqrt{2}\,.\,\mathrm{V}\,(\cosh u-\mu)^{\frac{1}{2}}\sum_{n=\infty}^{n=\infty}\frac{\sinh (n+\frac{1}{2})(u-u_2)}{\sinh (n+\frac{1}{2})(u_1-u_2)}e^{-(n+\frac{1}{2})u_1}\,\mathrm{P}_n(\mu).$$

If σ_1, σ_2 denote the surface densities of the charges on the spheres u_1, u_2, respectively, we have, by Coulomb's theorem,

$$\begin{aligned}4\pi\sigma_1 &= -\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial s_v} = \frac{\cosh u_1-\cos v}{a}\left.\frac{\partial\phi}{\partial u}\right|_{u=u_1}\\ &= \frac{\mathrm{V}}{\sqrt{2}\,.\,a}(\cosh u_1-\mu)\sum_{n=0}^{n=\infty} e^{-(n+\frac{1}{2})u_1}\left\{\frac{\sinh u_1}{\sqrt{(\cosh u_1-\mu)}}\right.\\ &\qquad\left. +(2n+1)\,\sqrt{(\cosh u_1-\mu)}\,\coth (n+\tfrac{1}{2})(u_1-u_2)\right\}\mathrm{P}_n(\mu),\end{aligned}$$

$$\begin{aligned}4\pi\sigma_2 &= \frac{\cosh u_2-\cos v}{a}\left.\frac{\partial\phi}{\partial u}\right|_{u=u_2}\\ &= \frac{\mathrm{V}}{\sqrt{2}\,.\,a}(\cosh u_2-\mu)^{\frac{3}{2}}\sum_{n=0}^{n=\infty}\frac{(2n+1)\,e^{-(n+\frac{1}{2})u_1}}{\sinh (n+\frac{1}{2})(u_1-u_2)}\mathrm{P}_n(\mu).\end{aligned}$$

Using Maxwell's notation we will denote the coefficients of capacity by

q_{11}, q_{22} and the coefficient of induction by q_{12}. To obtain these we put $V = 1$ and integrate the surface density over the appropriate sphere. The following integrals will be required,

$$\int_{-1}^{+1} \frac{P_n(\mu)}{(\cosh u - \mu)^{\frac{1}{2}}} d\mu = \frac{2\sqrt{2}}{2n+1} e^{-(n+\frac{1}{2})|u|}$$

and
$$\int_{-1}^{+1} \frac{P_n(\mu)}{(\cosh u - \mu)^{\frac{3}{2}}} d\mu = 2\sqrt{2} \frac{e^{-(n+\frac{1}{2})|u|}}{\sinh |u|}.$$

$$\begin{aligned} q_{11} &= 2\pi \int \rho\sigma_1 \, ds_u \\ &= \frac{a}{2\sqrt{2}} \int_{-1}^{+1} \sum_{n=0}^{n=\infty} e^{-(n+\frac{1}{2})u_1} \left\{ \frac{\sinh u_1}{(\cosh u_1 - \mu)^{\frac{3}{2}}} \right. \\ &\qquad \left. + (2n+1) \frac{\coth (n+\frac{1}{2})(u_1 - u_2)}{(\cosh u_1 - \mu)^{\frac{1}{2}}} \right\} P_n(\mu) \, d\mu \\ &= a \sum_{n=0}^{n=\infty} e^{-(2n+1)u_1} [1 + \coth (n+\tfrac{1}{2})(u_1 - u_2)] \\ &= 2a \sum_{n=0}^{n=\infty} \{ e^{(2n+1)u_1} - e^{(2n+1)u_2} \}^{-1}. \end{aligned}$$

$$\begin{aligned} q_{12} = q_{21} &= 2\pi \int \rho\, \sigma_2 \, ds_u \\ &= -\frac{1}{2\sqrt{2} \,.\, a} \int_{-1}^{+1} \sum_{n=0}^{n=\infty} \frac{(2n+1)\, e^{-(n+\frac{1}{2})u_1}}{(\cosh u_2 - \mu)^{\frac{1}{2}} \sinh (n+\frac{1}{2})(u_1 - u_2)} P_n(\mu) \, d\mu \\ &= -a \sum_{n=0}^{n=\infty} \frac{e^{-(n+\frac{1}{2})(u_1-u_2)}}{\sinh (n+\frac{1}{2})(u_1 - u_2)}, \qquad \text{if } u_2 < 0, \\ &= -2a \sum_{n=0}^{n=\infty} \{ e^{(2n+1)(u_1-u_2)} - 1 \}^{-1}. \end{aligned}$$

From considerations of symmetry it is clear that we should obtain for the third coefficient

$$q_{22} = 2a \sum_{n=0}^{n=\infty} \{ e^{-(2n+1)u_2} - e^{-(2n+1)u_1} \}^{-1}. *$$

* These expressions are not those obtained by Maxwell ('Electricity and Magnetism 3rd ed., vol. 1, p. 271), but it is possible to transform them into the forms there given, *e.g.*—

$$\begin{aligned} q_{11} &= 2a \sum_{n=0}^{n=\infty} [e^{(2n+1)u_1} - e^{(2n+1)u_2}]^{-1} \\ &= 2a \sum_{n=0}^{n=\infty} \sum_{r=0}^{r=\infty} e^{-(r+1)(2n+1)u_1 + r(2n+1)u_2}. \end{aligned}$$

Since $u_2 < 0 < u_1$ the double series is absolutely convergent, and we may interchange the order of summation

$$\begin{aligned} &= 2a \sum_{n=0}^{n=\infty} \sum_{r=0}^{r=\infty} e^{-(2r+1)(n+1)u_1 + (2r+1)nu_2} \\ &= a \sum_{n=0}^{n=\infty} \operatorname{cosech} [n(u_1 - u_2) + u_1], \end{aligned}$$

which is Maxwell's form.

When $u_2 > 0$ one sphere is enclosed within the other and there are slight differences in the coefficients due to the different forms taken by the integrals in these circumstances. We obtain

$$q_{11} = -q_{12} = -q_{21} = 2a \sum_{n=0}^{n=\infty} \{e^{(2n+1)u_1} - e^{(2n+1)u_2}\}^{-1},$$

$$q_{22} = \frac{a}{\sinh u_2} + 2a \sum_{n=0}^{n=\infty} \{e^{(2n+1)u_1} - e^{(2n+1)u_2}\}^{-1}.$$

For two equal spheres $u_2 = -u_1$.

$$q_{11} = q_{22} = a \sum_{n=0}^{n=\infty} \operatorname{cosech}(2n+1)\, u_1,$$

$$q_{12} = -a \sum_{n=0}^{n=\infty} e^{-(2n+1)u_1} \operatorname{cosech}(2n+1)\, u_1.$$

In the case of a sphere in the presence of an infinite plane, $u_2 = 0$.

$$q_{11} = -q_{12} = 2a \sum_{n=0}^{n=a} \{e^{(2n+1)u_1} - 1\}^{-1}.$$

We have tabulated below some of the more important of these functions.

Table I.—Capacity of Two Equal Spheres.

u.	$\frac{2r}{d} = \frac{\text{diameter of spheres}}{\text{centre distance}}$.	$\frac{q_{11}}{r}$.
0 ·2	0 ·9804	1 ·8368
0 ·4	0 ·9268	1 ·4796
0 ·6	0 ·8442	1 ·3055
0 ·8	0 ·7526	1 ·2011
1 ·0	0 ·6508	1 ·1284
1 ·2	0 ·5526	1 ·0916
1 ·4	0 ·4654	1 ·0609
1 ·6	0 ·3888	1 ·0408
1 ·8	0 ·3224	1 ·0273
2 ·0	0 ·2662	1 ·0183
2 ·5	0 ·1630	1 ·0067
3 ·0	0 ·0996	1 ·0025

Table II.—Capacity of a Sphere in the Presence of an Infinite Conducting Plane.

u_1.	$\frac{r}{p} = \frac{\text{radius of sphere}}{\text{distance of centre from plane}}$.	$\frac{q_{11}}{r}$.
0·2	0·9804	2·8994
0·4	0·9268	2·2367
0·6	0·8442	1·8982
0·8	0·7526	1·6679
1·0	0·6508	1·5093
1·2	0·5526	1·3942
1·4	0·4654	1·3083
1·6	0·3888	1·2429
1·8	0·3224	1·1915
2·0	0·2662	1·1434
2·5	0·1630	1·0887
3·0	0·0996	1·0525

In a future paper we will show how the method can be applied to the problems of the motion of two spheres in a perfect fluid.

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